With the pressure determined, the remaining unknown functions can also be found in their closed form. For example, the form of the diffraction shock wave AB can be computed from (1.1) according to the formula

$$\psi(y) = \frac{(\gamma+1)(y-k_1)}{4kk_1}E_1 + \frac{\psi(k_1)}{k_1}y - \frac{\gamma+1}{4k}y\int_{k_1}^{y} s^{-2}p(s)\,ds$$

where $\psi(k_1)$ is known from the solution of the problem in region 4.

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ON THE THEORY OF ELECTROMAGNETIC WAVE DIFFRACTION IN ACTIVE MEDIA

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The most effective methods of generation and intensification of electromagnetic waves are based on the interaction of beams of charged particles with attenuating media, in which use is made of the fundamental effect of a charged particle stream on the medium properties, with the latter changing from a passive state absorbing radiation to an active one intensifying the electromagnetic field. In particular, the essential difference between a passive and an active medium is confirmed by the fact that theorems related to fluctuating dissipation applicable to absorbing media do not hold in the case of active ones. Hence, it is to be expected that the electromagnetic wave diffraction in active media will also take place in a different manner.

Growth of the magnetic wave amplitude in an active medium is in fact limited by either nonlinear effects, or by the finite length of the system active section. In the latter case the change of the medium dielectric properties at the active section boundary must be accompanied by a transformation of the intensified wave into some other kind of waves. In particular, the problem of electromagnetic energy extraction from the system of particle beam-attenuating medium reduces to that of diffraction along the matching element. Therefore an investigation of the theory of diffraction in active media in not only of a theoretical, but also practical interest.

One of the problems of this kind, viz. the matching to a coaxial line of plasma waveguide in which electromagnetic waves are amplified by a beam of charged particles, is considered below. A coaxial guide should prove more effective in comparison with the usually utilized helical guide, particularly with high energy beams when the phase velocities of waves induced by the beam are close to the velocity of light in vacuum.

1. We shall assume that the waveguide is in the form of a cylinder ($0 \leqslant r \leqslant a$, $-\infty < z < +\infty$) filled by homogeneous plasma of density n_p, and placed in a strong magnetic field H_0 parallel to the cylinder axis ($\omega_H \gg \omega_p$, ω, where $\omega_H \equiv eH_0/mc$ is the gyroscopic frequency of an electron, e and m are respectively the charge and the mass, $\omega_p \equiv (4\pi e^2 n_p/m)^{1/2}$ is the plasma frequency, ω is the working frequency, and c is the velocity of light). A beam of particles of density n_b moves through the plasma waveguide at velocity V_b in the direction of $z > 0$. (Here and in the following subscripts p and b relate to the plasma and the beam parameters respectively). The coaxial waveguide consists of an infinitely thin semi-bounded ($z > 0$) conducting cylinder of radius a and of a conducting casing common to both waveguides ($r = b$, $-\infty < z < +\infty$, $b > a$).

The fields generated in this system by dissipation of one of the intensified waves in the plasma waveguide are to be determined.

The dependence of the wave number $k_=$ on frequency ω in the plasma waveguide ($z < 0$) is defined by the dispersion equation of the latter which is of the form [1]

$$D(k_=) \equiv \frac{\Delta_1(a)}{\Delta_0(a)} - \frac{k_\perp}{v}\frac{J_1(k_\perp a)}{J_0(k_\perp a)} = 0 \tag{1.1}$$

where

$$v(k_=) \equiv (k_=^2 - k^2)^{1/2}, \; k_\perp^2 \equiv -\varepsilon_=(k_=, \omega)v^2$$

$$k \equiv \frac{\omega}{c}, \quad \mathrm{Im}\,\omega > 0, \quad \mathrm{Re}\,v(k_=) > 0$$

$$\Delta_n(r) \equiv K_n(vr) I_0(vb) - (-1)^n I_n(vr) K_0(vb)$$

and $\varepsilon_=(k_=, \omega)$ is the longitudinal component of the system plasma-beam dielectric permeability tensor. In the particular case of weak spatial dispersion in which the thermal motion of beam particles may be neglected, we have [2]

$$\varepsilon_=(k_=, \omega) \equiv 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - k_= V_b)^2} \qquad \left(\omega_b^2 \equiv \frac{4\pi n_b e^2}{m}\right) \tag{1.2}$$

It will be realily seen that the solutions of (1.1) are of the form

$$k_\perp a = \lambda_n \tag{1.3}$$

where the slow changing functions of parameters of system λ_n are derived by substituting (1.3) into (1.1). In particular, the equation of λ_n for $kb < 1$ is of the form [1]

$$J_0(\lambda_n) = \lambda_n J_1(\lambda_n)\eta, \qquad \eta \equiv \ln(b/a) \tag{1.4}$$

For $\eta = 1$ we have $\lambda_1 = 1.25$, $\lambda_2 = 4.08$ etc.

It can be shown by substituting roots b of (1.4) into (1.3) that when the beam currents are weak, i.e. when $n_b \ll n_p$, the solutions γ_n of Eq. (1.1) dependent on the presence of the beam can be presented in the form

$$\gamma_n = \frac{\omega}{V_b} \pm \frac{\omega_b}{V_b \sqrt{\varepsilon_p^{(n)}}}, \quad \varepsilon_p^{(n)} \equiv 1 - \frac{\omega_p^2}{\omega^2} + \frac{\lambda_n^2 V_b^2}{\omega^2 a^2} \tag{1.5}$$

Such waves, also called charge density waves, are intensified in plasma at $\mathrm{Re}\, \varepsilon_p^{(n} < 0$. For $|\varepsilon_p^{(n)}| \to 0$ when the beam velocity V_b is close to the phase velocity V_φ of one of the plasma waveguide waves in the absence of a beam, the solution of the dispersion Eq. (1.3) is of the form

$$\left(\gamma_n - \frac{\omega}{V_b}\right)^3 = -\frac{\omega_b^2 \lambda_n^2}{2V_b \cdot \omega a^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-2} \tag{1.6}$$

$$V_\varphi^{(n)} \equiv c\left[1 + \frac{\lambda_n^2 c^2}{(\omega_p^2 - \omega^2)\, a^2}\right]^{-1/2} = V_b$$

The dispersion equation of the plasma cyclinder which fills the inner coaxial conductor may be obtained from (1.1) by passing to the limit $b \to a$, and its solutions are of the form of (1.3)–(1.6), on condition that the roots η_n of Eq. $J_0(\eta_n) = 0$ are substituted for constants λ_n.

With fixed parameters of this system the condition of intensification of longitudinal waves $\mathrm{Re}\,\varepsilon_p^{(\bar{n})} < 0$ can only be fulfilled for a finite number of waves in the left- and right-hand waveguides containing plasma. It follows from the obvious inequality

$$\lambda_n < \eta_n < \lambda_{n+1} < \eta_{n+1} \tag{1.7}$$

that the number n of intebsified waves in the left-hand waveguide is always greater than, or equal to the number m of intensified waves in the right-hand waveguide ($m \leqslant n \leqslant m + 1$). The methods of solving the problem in these two cases are different, and will be considered separately.

2. If condition

$$\lambda_1^2 < \frac{(\omega_p^2 - \omega^2)\, a^2}{V_b^2} < \eta_1^2 \tag{2.1}$$

is fulfilled, then there is no intensification in the coaxial guide, while in the left-hand waveguide there is only one intensified wave defined by the transverse wave number λ_1/a and the longitudinal wave number γ_1 ($\mathrm{Im}\ \gamma_1 < 0$)

$$E_{z1}(r,z) = E_1 J_0(\lambda_1 r/a) \exp(i\gamma_1 z) \tag{2.2}$$

We shall assume the amplitude E_1 of this wave as given, and shall determine the fields generated by its dispersion at the entry to the coaxial guide.

The problem reduces to finding the solution of the wave equation

$$\frac{1}{r}\frac{\partial}{r} r \frac{\partial E_z}{r} + \left(\frac{\partial^2}{\partial z^2} + k^2\right)\varepsilon_= * E_z = 0 \tag{2.3}$$

Here the asterisk denotes the multiplication of the field Fourier component by corresponding values of function $\varepsilon_=(k_=, \omega)$ defined by (1.2). Boundary conditions amount to stipulating the continuity of E_z and E_z' along the plasma surface ($r = a$, $z < 0$).

At the surface of conductors ($r = b, -\infty < z < +\infty$ and $r = a, z > 0$) a further condition must be satisfied, viz. $E_z = 0$.

Near the boundary of the inner coaxial conductor

$$\rho \equiv [(r-a)^2 + z^2]^{1/2} \to 0$$

the field energy must be finite because of the assumption of absence of radiation sources in this region [3]. When condition (2.1) is satisfied, all fields of the considered system decrease at great distances from the inhomogeneity, and the solution of Eq. (2.3) may be sought in the form of a superposition of plane waves in each of the regions in which the dielectric properties of the system remain continuous

$$E_z(r, z) = \int_{-\infty}^{+\infty} e_1(t) J_0[k_\perp(t) r] \exp(itz)\, dt$$

$$H_\varphi(r, z) = ik \int_{-\infty}^{+\infty} k_\perp(t) v^{-2}(t) e_1(t) J_1[k_\perp(t) r] \exp(itz)\, dt \qquad (2.4)$$

$$0 \leqslant r \leqslant a, \quad -\infty < z < +\infty$$

$$E_z(r, z) = \int_{-\infty}^{+\infty} e_2(t) \Delta_0(r, t) \exp(itz)\, dt$$

$$H_\varphi(r, z) = ik \int_{-\infty}^{+\infty} v^{-1}(t) e_2(t) \Delta_1(r, t) \exp(itz) dt$$

$$a < r < b, \quad -\infty < z < +\infty$$

$$v(t) \equiv (t^2 - k^2)^{1/2}, \quad \mathrm{Re}\, v(t) > 0, \quad k_\perp^2(t) \equiv -\varepsilon_=(k_=, \omega)\, v^2(t)$$

Here $e_1(t)$ and $e_2(t)$ are the unknown amplitudes of the Fourier fields which are to be determined from the boundary conditions. Functions $\Delta_m(r, t)$ are defined by (1.1).

We substitute fields (2.4) into the boundary conditions

$$\begin{aligned} E_z(r = a - 0, z < 0) &= E_z(r = a + 0, z < 0) \\ H_\varphi(r = a - 0, z < 0) &= H_\varphi(r = a + 0, z < 0) \\ E_z(r = a - 0, z < 0) &= E_z(r = a + 0, z < 0) = 0 \end{aligned} \qquad (2.5)$$

The systems of paired integral equations thus obtained may by the lemma of Wiener-Paley-Rappoport [4, 5] be reduced to the following system of Hilbert boundary value problems [6, 7] along axis $\mathrm{Im}\, t = 0$:

$$\varphi^+(t) - \psi^+(t) = \xi^-(t) \qquad (2.6)$$

$$\frac{ik}{v}\left[\frac{\Delta_1(a, t)}{\Delta_0(a, t)} \psi^+(t) - \frac{k_\perp(t)}{v} \frac{J_1(k_\perp a)}{J_0(k_\perp a)} \varphi^+(t)\right] = \varkappa^-(t) \qquad (2.7)$$

Solution (2.6), (2.7) makes it possible to determine $e_1(t)$ and $e_2(t)$ using the following relations:

$$e_1(t) = \frac{\varphi^+(t)}{J_0(k_\perp a)}, \qquad e_2(t) = \frac{\psi^+(t)}{\Delta_0(a, t)} \qquad (2.8)$$

From the condition of boundedness of the energy in proximity of the inner coaxial conductor boundary [3] follows that the polynomial $P(t)$ in the general solution (2.6)

$$\varphi^+(t) - \psi^+(t) = P(t) = \xi^-(t)$$

is zero. The boundary value problem (2.7) is then reduced to

$$\Phi_1^+(t) = G_0(t)\,\Phi_1^-(t) \tag{2.9}$$

where the following notation has been used:

$$\Phi_1^+(t) \equiv \frac{ik(1+in_=)}{(t+k)^{1/2}}\,\varphi^+(t), \quad \Phi_1^-(t) \equiv \varkappa^-(t)(t-k)^{1/2}$$
$$G_0(t) \equiv (1+in_=)D^{-1}(t), \quad n_= \equiv \left(\frac{\omega_p^2}{\omega^2}-1\right)^{1/2}, \ \mathrm{Re}\, n_= > 0 \tag{2.10}$$

The solution of the homogeneous boundary value problem (2.9) vanishing at infinity is uniquely defined by its index [6, 7] which essentially depends on the presence of intensification in the left-hand waveguide. If absorption is considerable and the beam density low, then intensification is absent ($\mathrm{Im}\ \gamma_1 < 0$). In that case the index of problem (2.9) is zero (see Appendix), hence its solution is uniquely and identically zero. This means that in the absence of intensification the dispersed fields cannot occur in the system without an external inducing field which can be any superposition of the considered waveguides own waves arriving form infinity.

When absorption is low and the beam density sufficiently great, then intensification takes place ($\mathrm{Im}\ \gamma_1 < 0$). The problem index is in this case equal unity (see Appendix), so that there exists a solution of the problem which vanishes at infinity and depends on the single constant

$$\Phi_1^+(t) = \frac{X_1^+(t)\,C}{t-\Gamma_+}, \qquad \Phi_1^-(t) = \frac{X_1^-(t)\,C}{t-\Gamma_-} \tag{2.11}$$

Here $\Gamma_\pm$ are arbitrary constants satisfying conditions $\mathrm{Sgn}\ \mathrm{Im}\ \Gamma_\pm = \pm 1$, and functions $X_1^\pm(t)$ are bounded at infinity solutions of the homogeneous problem of conjugation

$$X_1^+(t) = G_1(t)\ X_1^-(t)$$
$$G_1(t) \equiv G_0(t)\,\frac{t-\Gamma_-}{t-\Gamma_+} \tag{2.12}$$

Constant C in solution (2.11) is uniquely defined by the stipulation that the intensified wave amplitude (the pre-exponential factor in (2.4)) be equal to the known magnitude E_1 (see (2.2)). Substituting $e_1(t)$ from (2.8) and (2.11) into (2.4) we obtain

$$C = -\frac{k}{2\pi}E_1\frac{J_0(\lambda_1)}{X_1^-(\gamma_1)}, \qquad D'(\gamma_1)\frac{\gamma_1-\Gamma_+}{(\gamma_1+k)^{1/2}} \tag{2.13}$$
$$\Gamma_- = \gamma_1, \qquad D' \equiv \frac{d}{dt}D(t)$$

The final expression of the Fourier components of the dispersed fields is derived using (2.8)–(2.13)

$$e_1(t) = \frac{(t+k)^{1/2}}{ik(1+in_=)}\,\frac{C}{J_0(k_\perp a)}\,\frac{X_1^+(t)}{t-\gamma_1} \tag{2.14}$$
$$e_2(t) = \frac{J_0(k_\perp a)}{\Delta_0(a,t)}\,e_1(t)$$

3. We shall now consider solutions in the case in which the following inequality is satisfied:

$$\eta_1^2 < \frac{(\omega_p^2-\omega^2)\,a^2}{V_b^2} < \lambda_2^2 \tag{3.1}$$

so that intensification takes place also inside of the coaxial conductor (in these conditions only the first radial harmonics of lowest radial wave numbers can be intensified in both parts of the plasma waveguide). In this case in region $0 \leqslant r \leqslant a$, $z > 0$ the field does not decrease with increasing distance from the inhomogeneity, hence, representation (2.4) is inapplicable. It is however possible to separate from the field of the right-hand waveguide an intensified wave, and then represent the remaining field in the form of plane wave superposition

$$E_z(r, z) = A_1 J_0(\eta_1 r/a) \exp(i\varkappa_1 z) + \int_{-\infty}^{+\infty} f_1(t) J_0[k_\perp(t) r] \exp(itz)\, dt \quad (3.2)$$

$$0 \leqslant r \leqslant a, \quad -\infty < z < +\infty$$

$$E_z(r, z) = \int_{-\infty}^{+\infty} f_2(t) \Delta_0(r, t) \exp(itz)\, dt \quad (3.3)$$

$$a \leqslant r \leqslant b, \qquad -\infty < z < +\infty$$

Here $\eta_1 = 2{,}405, \ldots$, $\varkappa_1$ is defined by expressions (1.5) or (1.6) (with the substitution $\gamma_1 \to \varkappa_1$, $\lambda_1 \to \eta_1$); constant A_1 and the unknown amplitudes $f_1(t)$ and $f_2(t)$ are determined from the boundary condition (2.5).

Substituting into the boundary conditions (2.5) the fields E_z from (3.2) and (3.3) and fields H_φ defined by relation

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) H_\varphi = ik \frac{\partial E_z}{\partial r}$$

we obtain the nonhomogeneous boundary value problem

$$\Phi_0^+(t) = G_0(t) \Phi_0^-(t) + \frac{A}{2\pi i} G_0(t) \frac{(t-k)^{1/2}}{t - \varkappa_\perp}$$

$$A \equiv A_1 \frac{ik\eta_1 J_1(\eta_1)}{(\varkappa_1^2 - k^2)\, a} \quad (3.4)$$

of the type of (2.9) in which function $G_0(t)$ is as previously defined by (2.10), and the Fourier amplitudes $f_1(t)$ and $f_2(t)$ are expressed by the solution of (3.4) using the following relations:

$$f_1(t) \equiv \frac{(t+k)^{1/2} \Phi_0^+(t)}{ikJ_0(k_\perp a)(1 + in_=)}, \quad f_2(t) \equiv \frac{J_0(k_\perp a)}{\Delta_0(a, t)} f_1(t) \quad (3.5)$$

It can be shown (see Appendix) that the index of problem (3.4) is in this case zero, hence, its solution valishing at infinity does exist and is unique. Constant A_1, so far considered as known, is uniquely defined as in the previous case by the requirement for the amplitude of the intensified wave field E_z to be at $z \to -\infty$ equal to the known magnitude E_1 in accordance with (2.2)

$$A_1 = \frac{(\varkappa_1 - \gamma_1)(\varkappa_1^2 - k^2)\, a}{\eta_1 [(\gamma_1 + k)(\varkappa_1 - k)]^{1/2}} \frac{X_0^-(\varkappa_1)}{X_0^-(\gamma_1)} E_1 \quad (3.6)$$

where $X_0^\pm(t)$ are the bounded at infinity solutions of the homogeneous conjugate problem $X_0^+ = G_0 X_0^-$. Thus, in the case of (3.1), the solution is uniquely defined by expres sions (3.4)–(3.6).

4. The method of solution presented above can be extended to the case of an arbitrary number of waves intensified in the left-hand (n) and the right-hand (m) waveguides ($1 \leqslant m \leqslant n \leqslant m + 1$).

In this case the field in region $0 \leqslant r \leqslant a$, $z > 0$ is to be sought in the form (cf. (3.2))

$$E_z = \sum_{l=1}^{m} A_l J_0 (\eta_l r / a) \exp (i\varkappa_l z) + E_\infty \tag{4.1}$$

where the sum corresponds to waves intensified in the coaxial conductor, and E_∞ to fields decreasing at considerable distances from the inhomogeneity, and can be represented in the form of superposition of plane waves.

The unknown constants A_l for $n = m$, when the index of a boundary value problem of the kind of (3.4) is zero, is determined in the same way as A_1 in (3.6) from the known amplitudes E_e of the intensified waves of the left-hand waveguide

$$E_{zl} = E_l J_0 (\lambda_l r / a) \ \exp \ (i\gamma_l z) \tag{4.2}$$

For $n = m + 1$ the number of known constants in (4.2), and consequently the number of equations required for the determination of the unknown A_l is equal to $m + 1$, i.e. it exceeds the number of unknowns by one. In this case, however, a homogeneous problem of the kind of (2.9) has a nonzero solution containing (similarly to (2.11)) one more unknown constant, so that the number of equations remains, as previously equal to the number of unknowns.

5. The problem of energy input into the coaxial conductor in the presence of a weak magnetic field $(\omega_H \ll \omega, \ \omega_p, \ \omega - k_= V_b)$ may be considered in the same manner. In this case the dispersion equation of the plasma waveguide with a beam is of a form similar to (1.1), viz.

$$\varepsilon_= (k_=, \ \omega) \ \frac{I_1 (k_\perp a)}{k_\perp a I_0 (k_\perp a)} = \frac{\Delta_1 (a)}{v \Delta_0 (a)} \tag{5.1}$$

$$k_\perp^2 \equiv - \frac{\varepsilon_= (k_=, \ \omega) \, [k_=^2 + (\omega_p^2 + \omega_b^2 - \omega^2) \ c^{-2}]}{\varepsilon_= \ (k_=, \ \omega) + \beta^2 \varepsilon_p \, [\varepsilon_p - \varepsilon_= \ (k_=, \ \omega)]}$$

$$\beta \equiv V_b / c, \qquad \varepsilon_p \equiv 1 - \frac{\omega_p^2}{\omega^2}$$

The analysis of this equation shows that in the absence of a beam ($n_b = 0$) only a single slow ($V_\Phi < c$) surface wave ($k_\perp^2 < 0$) can propagate in the plasma waveguide in the $0 < \omega < 2^{-1/2}\omega_p$ frequency range. The introduction of a beam into such a waveguide results in the appearance of slow volumetric waves (called charge density waves, or longitudinal waves) the longitudinal wave numbers of which for $n_b \ll n_p$ are defined by relations

$$\gamma_n V_b = \omega \pm \frac{\omega_b}{\sqrt{\varepsilon_p}} \left\{ 1 + \beta^2 \lambda_n^2 \varepsilon_p \left[\frac{\omega^2 a^2}{V_b^2} + \lambda_n^2 + (\omega_b^2 + \omega_p^2 - \omega^2) \ \frac{a^2}{c^2} \right]^{-1} \right\}^{1/2} \tag{5.2}$$

where constants λ_n (or $\varkappa_n$) may be derived after substitution of (5.2) into (5.1).

It follows from (5.2) that in the case of the weak magnetic field considered here the intensification condition $\omega < \omega_p$ is independent of the radial wave number. Hence, for $\omega < \omega_p$ all charge density waves of the right-hand waveguide are intensified. It will be readily seen that the method of determination of dispersed fields is in this case analogous to that adduced in Sect. 4 for a finite number of intensified waves in a waveguide in the presence of a strong longitudinal magnetic field. The only difference is in that in the sum in (4.1), $m = \infty$ is to be specified in this case. Consequently the number of equations of the system defining constants A_l must be infinitely great, due to the number of intensified waves in the left-hand waveguide, the amplitudes of which (E_l in

(4.2)) are known, being equal to the number of intensified waves in the right-hand side waveguide. The index of the corresponding homogeneous problem is zero for $2^{-1/2}\omega_p < \omega < \omega_p$, when the surface wave is not subject to intensification, and is equal plus unity for $\omega < 2^{-1/2}\omega_p$, when the surface wave is intensified (see Appendix). In the latter case the number of unknowns, and correspondingly the number of equations are increased by one.

6. When the Fourier components of dispersed field, e.g. (2.14) and (2.15) are known, then these fields themselves can be determined. The determination of conditions in which the effect of the beam on the diffraction characteristics of the problem is at its maximum, is of considerable interest.

It should be first of all noted that the charge density waves defined by (1.5) cannot exist in a waveguide in the absence of a beam, even when $n_b \to 0$. Hence, the analysis presented above is necessary for an investigation of such waves.

A comparison of the diffraction characteristics of passive and active systems is possible in the case of resonance only, when at low beam density its velocity is close to the phase velocity of one of the proper waves of the plasma waveguide without a beam (see (1.6)). The transformation coefficients defined as the ratio of magnetic amplitudes of the coaxial ($k_= = k$, $r = a + 0$, $z > 0$) and the incident ($k_= = \gamma_1$, $r = a - 0$, $z < 0$) waves, coincide to within magnitudes of the order of the intensification coefficient ($|\gamma_n - \omega V_b^{-1}| \ll \gamma_n$). In fact the computation of the coaxial wave amplitude using (2.14) and (2.4) yields in the case of an active system

$$H_+^{(k)} = 2\pi i \operatorname*{res}_{t=k} \frac{ike_2(t)\,\Delta_1(a,t)}{v(t)} = -\left(\frac{\gamma_1+k}{2k}\right)^{1/2} \frac{X_1^+(k)}{X_1^-(\gamma_1)} \frac{D'(\gamma_1)(\gamma_1-\Gamma_+)}{1+in_=} H_1 \quad (6.1)$$

where H_1 is the amplitude of the intensified wave magnetic field.

The expression defining the amplitude of a coaxial wave generated by the plasma waveguide own wave coming from $z = -\infty$ in the absence of a beam ($\mathrm{Im}\ \gamma_1 > 0$) may be found from (3.2)–(3.6) by substituting $A_1 \to E_1$ and $\varkappa_1 \to \gamma_1$. After simple computations we derive for the case of a passive system

$$H_-^{(k)} = \left(\frac{\gamma_1+k}{2k}\right)^{1/2} \frac{X_n^+(k)}{X_n^-(\gamma)} H_1, \quad \gamma \equiv \gamma_1^* \quad (6.2)$$

Here H_1 is the amplitude of the incident wave magnetic field, and $X_n^{\pm}(t)$ are the solutions of the bounded at infinity homogeneous problem of conjugation

$$X_n^+(t) = G_n(t), \ \ X_n^-(t), \ \ G_n(t) \equiv G_0(t)\,|_{n_b=0} \quad (6.3)$$

The computation of integrals appearing in (6.1) shows that for $n_b \to 0$, when the intensification coefficient $\alpha \equiv |\mathrm{Im}\ \gamma_1| \ll \mathrm{Re}\ \gamma_1 = \gamma_0$

is small, the following equations hold

$$X_1^-(\gamma_1) = -X_n^-(\gamma) \frac{D'(\gamma_1)(\gamma_1-\Gamma_+)}{1+in_-}$$

so that Expressions (6.1) and (6.2) coincide in this case to within magnitudes of the order of α/γ_0.

The physical meaning of this result may be interpreted as follows. The transformation coefficients are defined in the general case by the field pattern close to the inhomogeneity. Because at weak intensification the field pattern in the case of resonance of an active waveguide differs from that of this wave in a passive waveguide only by magnitudes of the order of intensification along the wavelength, hence, the difference of diffraction characteristics will also be small.

Thus the above analysis is necessary in the case of high beam currents, when the intensification along the wavelength (α/γ_0) is not small, and also in investigations of the transformation of charge density waves. In the general case of an arbitrary beam current the coefficient of transformation of an intensified wave into a coaxial one is determined from Expression (6.1).

Appendix. Index ν of the boundary value problems (2.9) and (3.4) is defined by the increase of the logarithm of function $G_0(t)$ along the contour $\mathrm{Im}\, t = 0$ [6, 7]

$$\nu \equiv \frac{1}{2\pi i} \ln G_0(t) \Big|_{-\infty}^{+\infty} \tag{A.1}$$

The only singularities of $G_0(t)$ are zeros and poles. These singularities correspond to the waveguide own waves which make up the considered system. For $n_b \to 0$ function $G_0(t)$ is even, hence index ν is zero. In the presence of a beam $(n_b \neq 0)$ the number of zeros $t_m = \varkappa_m$ and poles $t_n = \gamma_n$ of function $G_0(t)$ corresponding to volumetric waves of the plasma waveguide $(k_\perp(\gamma_n a) = \lambda_n,\ k_\perp(\varkappa_m a) = \eta_m)$ is doubled owing to the additional roots of the kind of (1.5), i.e. due to the presence of charge density waves in the beam. If absorption $(\mathrm{Im}\,\omega)$ is high and the beam density low, then intensification is absent $(\mathrm{Im}\,\gamma_1 > 0)$ even with condition (2.1) fulfilled. In this case all zeros and poles corresponding to charge density waves lie close to point $t_0 \equiv \omega b/Vb$ in the upper half-plane of the complex variable t. Each time a root $\varkappa_m^{(\pm)}$ is bypassed the index ν increases, while the bypassing of each pole $\gamma_n^{(\pm)}$ leads to the index being halved. It can be shown by grouping in (A.1) terms corresponding to roots $\varkappa_m^{(\pm)}$ and poles $\gamma_m^{(\pm)}$ with the same numbers m that in this case index ν is zero, as the bypassing of each pair of these singular points does not result in an increase of the index.

If however the beam density is sufficiently high, then with condition (2.1) fulfilled we have $\mathrm{Im}\gamma_1^{(-)} < 0$, and the contributions of poles $\gamma_1^{(\mp)}$ to the index of (A.1) compensate each other. The corresponding pair of roots $\varkappa_1^{(\pm)}$ remains in accordance with the right-hand side of inequality (2.1) in the upper half-plane, hence the contribution of these to (A.1) is equal plus one. The remaining pairs of roots and poles do not contribute any increments to (A.1). Thus, with condition (2.2) satisfied, the index of problem (2.9) is equal plus one.

If, on the other hand, inequalities (3.1) are fulfilled, then both the roots $\varkappa_1^{(\pm)}$ and the poles $\gamma_1^{(\pm)}$ lie in different half-planes, and the index of problem (3.4) is zero, as in the case of absence of intensification.

By grouping poles and zeros of $G_0(t)$ corresponding to intensified and attenuated waves of the two waveguides it can be similarly shown that in the general case of an arbitrary number of intensified waves in the left-hand (n) and right-hand (m) waveguides the following equality holds

$$\nu = \begin{cases} 1 & m = n - 1 \\ 0 & m = n \end{cases} \tag{A.2}$$

It is readily seen that this equality remains valid also in the case of resonance (1.6), as one of the roots of (1.6) corresponds to the proper wave of the plasma waveguide without a beam, while only one of the other two corresponding to charge density waves may be an intensified one.

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ON COMPOSITE STEADY GRAVITATIONAL WAVES OF FINITE AMPLITUDE

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The problem of plane steady waves of finite amplitude generated by pressure periodically distributed over the surface of a heavy fluid of infinite depth was first formulated by Stretenskii in 1953 [1] who also gave its approximate solution. The exact solution of this problem was presented by the Author for an infinitely deep stream in papers [2, 3] and also for streams of finite depth in [4–6].

All of these papers had investigated waves which cease to exist when the periodic part of the pressure distributed over the stream surface vanishes and the flow becomes uniform. We shall call such waves induced. Waves of constant amplitude occurring at particular flow velocities under conditions of constant pressure over the whole surface will be called free waves. An exact solution of the problem of these waves was first given by Nekrasov in 1921 [7].

The possibility of a simultaneous occurrance of these two kinds of waves of small finite amplitude in a steam of infinite depth at particular flow velocities is established below. We shall call such waves composite waves. When the periodic part of the pressure distributed over the surface vanishes, such waves are transformed into free waves.

The general method of computation of characteristics of these waves is presented. The complete computation of the first three approximations, and an approximate equation of the wave profile are given.